

Statistical mechanics of the spherical hierarchical model with random fields

Fernando L. Metz^{1,2}, Jacopo Rocchi², Pierfrancesco Urbani³

¹Instituto de Física, Universidade Federal do Rio Grande do Sul, Caixa Postal 15051, 91501-970 Porto Alegre, Brazil

²Dipartimento di Fisica, Sapienza Università di Roma, P.le A. Moro 2, I-00185 Roma, Italy

³IPhT, CEA/DSM-CNRS/URA 2306, CEA Saclay, F-91191 Gif-sur-Yvette Cedex, France

E-mail: fmetzfmetz@gmail.com

Abstract. We study analytically the equilibrium properties of the spherical hierarchical model in the presence of random fields. The expression for the critical line separating a paramagnetic from a ferromagnetic phase is derived. The critical exponents characterising this phase transition are computed analytically and compared with those of the corresponding D -dimensional short-range model, leading to conclude that the usual mapping between one dimensional long-range models and D -dimensional short-range models holds exactly for this system, in contrast to models with Ising spins. Moreover, the critical exponents of the pure model and those of the random field model satisfy a relationship that mimics the dimensional reduction rule. The absence of a spin-glass phase is strongly supported by the local stability analysis of the replica symmetric saddle-point as well as by an independent computation of the free-energy using a renormalization-like approach. This latter result enlarges the class of random field models for which the spin-glass phase has been recently ruled out.

PACS numbers: 75.10.Nr, 64.60.F-, 64.60.ae

1. Introduction

Random field models are ferromagnetic systems of spins with a random field at each site. They have been introduced by Imry and Ma [1] and, despite many years of studies, their critical behaviour is still not completely understood, even in the simplest situation where the random fields are uncorrelated. The study of their critical behaviour is strictly related to a property called dimensional reduction, according to which the critical exponents of a random field model in D dimensions is equivalent to those of the corresponding pure model in $D - 2$ dimensions. This property has been deeply analysed using different methods [2, 3, 4], and it has been shown that dimensional reduction does not hold at low dimensions [5]. The reason for this behaviour is not entirely clear, and different scenarios have been proposed to explain the critical behaviour of random field models (see [6] and references therein).

Some years ago it has been suggested that the main reason for the breakdown of dimensional reduction could be the presence of an equilibrium spin-glass phase between a high temperature paramagnetic and a low temperature ferromagnetic phase [7]. However, this intermediate spin-glass phase has been rigorously ruled out in a large class of random field models [8, 9], which do present a breakdown of the dimensional reduction property. These kind of models are defined in terms of pairwise interacting spins placed on the vertices of arbitrary graphs, where spins are of the Ising type [8]. This result has been recently extended to the case of a scalar field theory [9].

Another interesting class of models consists of those in which the couplings are arranged in an hierarchical structure. The prototypical example is the Dyson hierarchical model [10], introduced a long time ago to study phase transitions in one-dimensional models with interactions decaying as a power-law of the inter-site distance. The Dyson hierarchical model has been intensively studied [11, 12, 13], mostly because of its remarkable properties from the renormalization group point of view, which can be shown to be deeply connected with the approximate recursion formula derived by Wilson [14, 15], allowing to efficiently compute critical exponents [13, 16].

One of the main reasons why the Dyson hierarchical model has attracted a lot of attention is that it can be used to investigate non-mean-field critical behaviour. In fact, short-range D -dimensional systems are well described by mean-field theory for D large enough and a particular dimension separates mean-field from non-mean-field critical behaviour. Dyson hierarchical models exhibit qualitatively a similar phenomenology, with the dimension D replaced by an exponent τ , the latter being responsible for controlling the power-law decay of the interactions as a function of the inter-site distance. Although this intuitive analogy has been recently exploited to study quenched disorder systems in the non-mean-field sector [17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29], it is not clear to which extent there is a clear mapping between the spatial dimension D and the exponent τ [25, 26, 27, 29]. Indeed, a strict mapping holds in the mean-field region, whereas it does not give satisfying results for low enough dimensions [27, 29].

Here we focus on a spherical version of the Dyson hierarchical model in the presence

of random fields, in which the spins are continuous variables and the phase space is constrained to the surface of a sphere. The ferromagnetic spherical model was introduced a long time ago by Berlin and Kac [30] and its hierarchical counterpart has been considered in references [31, 32]. The main motivation to study spherical models lies in their exactly solvable nature [33], which renders a full analytical study of the critical properties possible, even in the presence of quenched disorder. Although the random field spherical model with short-range interactions in D dimensions has been previously studied [34], the Dyson hierarchical version constitutes an interesting testing ground to address at once different issues, such as the dimensional reduction problem, the existence of a spin-glass phase and the aforementioned relationship between the critical properties of short-range and long-range systems. We point out that the D -dimensional short-range counterpart of our model always displays a non-zero Edwards-Anderson order-parameter [34]. This is probably not related to the existence of a spin-glass phase, but simply reflects the presence of non-zero local magnetisations due to the quenched random fields. A more refined analysis is certainly needed in order to probe the existence of spin-glass states in the spherical model with random fields.

In this work we perform a thorough study of the equilibrium properties of the spherical hierarchical model in the presence of random fields, showing that the system undergoes a phase transition between a paramagnetic and a ferromagnetic phase. Exact analytical results for the critical exponents are derived in the mean-field as well as in the non-mean-field regime. By comparing our results with those of references [31, 34], we show that there is an exact mapping between the critical exponents of the spherical hierarchical model and those of the corresponding D -dimensional short-range system. Contrary to the case of Ising spins [25, 26, 27, 29], such mapping is exact here, even in the non-mean-field regime. We also show that the critical exponents of the random field model and those of the corresponding pure model obey a certain relation, proposed in [27], which is analogous to the dimensional reduction property. In particular, this property holds for any value of τ , in contrast to the hierarchical model with Ising spins [27], where dimensional reduction breaks down in the non-mean-field regime. The free-energy of the spherical hierarchical model is computed exactly using two different methods: a recursive approach [13], based on the invariance of the Hamiltonian under a renormalization-like transformation, and the standard replica method [35]. Finally, we show that the replica symmetric saddle-point is locally stable in the whole phase diagram, which strongly supports the absence of a spin-glass phase.

The rest of the paper is organised as follows. In the next section we define the hierarchical spherical model with random fields, while in section 3 we explain how the free-energy and the equation of state for this model are derived using both the recursive method and the replica method. The phase diagram and the absence of a spin-glass phase are discussed in section 4. In section 5 we discuss the computation of the critical exponents and in section 6 we comment on these results. Some conclusions are presented in section 7.

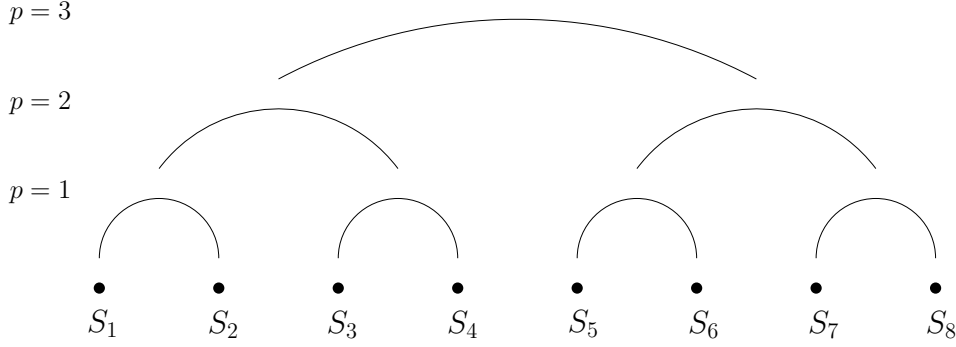


Figure 1. Pictorial representation of the interactions between spins in the hierarchical model defined by the Hamiltonian of eq. (3) with $n = 3$ levels. The coupling between S_i and S_j is defined by J_{ij} . For instance, the interactions between S_1 and the other spins are explicitly given by $J_{12} = b_1 + b_2 + b_3$, $J_{13} = J_{14} = b_2 + b_3$ and $J_{1,j} = b_3$, for $5 \leq j \leq 8$.

2. The spherical hierarchical model with random fields

We study a one-dimensional model composed of $N = 2^n$ real-valued spins $\{S_i\}_{i=1,\dots,N}$, with an external field $h_i = h + r_i$ acting on each spin S_i . The quantity h denotes the uniform part of the field, while $\{r_i\}_{i=1,\dots,N}$ are drawn independently from the Gaussian distribution

$$p(r) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{r^2}{2\sigma^2}\right). \quad (1)$$

The system is governed by the following Hamiltonian

$$\mathcal{H}(\mathbf{S}) = -\frac{1}{2} \sum_{i,j=1}^N J_{ij} S_i S_j - \sum_{i=1}^N h_i S_i, \quad (2)$$

where J_{ij} denotes the coupling between S_i and S_j . The set of all couplings $\{J_{ij}\}_{i,j=1,\dots,N}$ can be accommodated in the symmetric interaction matrix \mathbf{J} , with $J_{ii} = 0$ for $i = 1, \dots, N$.

Here we follow the original work of Dyson [10] and we consider a model where the couplings are organised in a hierarchical block structure. The hierarchy of interacting spins contains a total number of n levels, where $p = 1$ and $p = n$ are, respectively, the lowest and the highest level. The system is divided into 2^{n-p} distinct groups or blocks of spins at a certain level p , each group containing 2^p mutually interacting spins. The coupling between a pair of spins within a given block of level p is defined as b_p . These definitions lead naturally to a matrix \mathbf{J} where the off-diagonal elements are arranged in a block structure, with blocks of dimension $1, 2, 2^2, \dots, 2^{n-1}$. These matrix elements are given explicitly by $J_{ij} = \sum_{p=1}^k b_p$, where $k = 1 + \lfloor \log_2 |i - j| \rfloor$, and $\lfloor x \rfloor$ denotes the largest integer not greater than x . Substituting the explicit form of \mathbf{J} in eq. (2), the Hamiltonian reads

$$\mathcal{H}(\mathbf{S}) = -\frac{1}{2} \sum_{p=1}^n b_p \sum_{r=1}^{2^{n-p}} \left(\sum_{i=1}^{2^p} S_{(r-1)2^p+i} \right)^2 + \frac{A_n}{2} \sum_{i=1}^{2^n} S_i^2 - \sum_{i=1}^{2^n} h_i S_i, \quad (3)$$

where $b_p = 2^{-\tau p}$, with $\tau > 1$. As we will see below, the latter condition is required to obtain a bounded spectrum for the matrix \mathbf{J} in the limit $N \rightarrow \infty$. The term including $A_n = \sum_{p=1}^n b_p$ removes the self-interactions of the model. A schematic representation of the hierarchical block structure of the model is displayed on figure 1.

The coupling between two spins separated by a distance of $O(N)$ scales as $O(1/N^\tau)$, and the hierarchical model exhibits the same long-distance behaviour as a one-dimensional model with interactions decaying as a power-law of the inter-site distance. By varying the exponent τ , this class of one-dimensional models interpolates between systems with long-range or mean-field interactions and systems with short-range interactions, typical of models defined on finite-dimensional lattices. The hierarchical arrangement of the couplings has an extra advantage: the Hamiltonian preserves its structure under renormalization transformations, which usually leads to exact iterative methods of solution.

The matrix \mathbf{J} has $n + 1$ different eigenvalues given by

$$\lambda_p^{(n)} = \frac{2^{-\tau n} - 1}{2^\tau - 1} + \frac{1 - 2^{-(\tau-1)(p-1)}}{2^{\tau-1} - 1}, \quad p = 1, \dots, n + 1. \quad (4)$$

For $p < n + 1$, the eigenvalue $\lambda_p^{(n)}$ has a degeneracy factor of 2^{n-p} , which comes from the symmetry between the blocks at level p . The largest eigenvalue, obtained by setting $p = n + 1$ in eq. (4), has a degeneracy factor equal to 1. In the thermodynamic limit, we can introduce the spectral density

$$\rho(\lambda) = \sum_{p=1}^{\infty} 2^{-p} \delta(\lambda - \lambda_p), \quad (5)$$

where $\lambda_p = \lim_{n \rightarrow \infty} \lambda_p^{(n)}$. The spectrum of \mathbf{J} is clearly bounded provided $\tau > 1$, with the largest eigenvalue defined, for $n \rightarrow \infty$, as follows

$$\lambda_\infty = \lim_{p \rightarrow \infty} \lambda_p = \lambda_1 + \frac{1}{2^{\tau-1} - 1}, \quad (6)$$

with λ_∞ representing an accumulation point of the spectrum, since the difference $\lambda_{p+1} - \lambda_p$ vanishes exponentially as a function of p .

The partition function of the system in equilibrium at temperature $T = \beta^{-1}$ reads

$$\mathcal{Z}_N = \int d\mathbf{S} \delta(N - |\mathbf{S}|^2) \exp[-\beta \mathcal{H}(\mathbf{S})], \quad (7)$$

where $\mathbf{S} = (S_1, \dots, S_N)$ and $d\mathbf{S} = \prod_{i=1}^N dS_i$. The Dirac delta function imposes the spherical constraint by restricting the above integral to the global configurations that fulfill $\sum_{i=1}^N S_i^2 = N$. Due to this spherical constraint, an arbitrary real parameter a can be introduced in \mathcal{Z}_N , which allows us to rewrite eq. (7) as follows

$$\mathcal{Z}_N = \int d\mathbf{S} \int_{-\infty}^{\infty} \frac{ds}{2\pi} \exp[-\beta \mathcal{H}(\mathbf{S}) + (a + is)(N - |\mathbf{S}|^2)], \quad (8)$$

where we have used the Fourier integral representation of the Dirac delta function. As will be clearer below, the introduction of the regularizer a is convenient to guarantee the convergence of the integral over \mathbf{S} .

Our primary aim consists in computing the free-energy per spin in the thermodynamic limit

$$f = - \lim_{N \rightarrow \infty} \frac{1}{\beta N} \ln \mathcal{Z}_N, \quad (9)$$

from which we have access to the thermodynamics of the model and, eventually, to its critical behaviour.

3. Solution of the model

In this section we present two different methods to compute the partition function in the limit $N \rightarrow \infty$. The recursive method explores the hierarchical structure of the model and the trace over the spins is calculated iteratively by making a sequence of renormalization-like transformations on the partition function. This method is rather flexible, in the sense that it allows to calculate the trace over the spins for a single, finite instance of the random fields. In a subsequent stage, one needs to employ the saddle-point method to evaluate $\ln \mathcal{Z}_N$ for $N \rightarrow \infty$, and the statistical properties of the random fields become important.

The second approach is the well-known replica method [35], where the average of $\ln \mathcal{Z}_N$ over the random fields is computed by replicating the system q times. In the replica setting, the limit $q \rightarrow 0$ can be only performed by assuming a particular structure for the order-parameters. Here we show that the simplest assumption, namely the replica symmetric *ansatz*, yields the same averaged free-energy and the same equation of state as obtained through the recursive approach, provided the distribution of fields $p(r)$ is given by eq. (1).

3.1. The recursive method

In this section we show how to compute the trace over the spins using a simple change of integration variables combined with the model hierarchical structure. Such approach has been employed to study hierarchical models in the context of interacting spin systems (see [36] and references therein) and Anderson localisation [37, 38, 39].

By substituting eq. (3) in eq. (8) and choosing a sufficiently large, we assure that the integral over \mathbf{S} in eq. (8) is convergent, which allows us to interchange the order of the $d\mathbf{S}$ and ds integrations

$$\mathcal{Z}_N = \int_{-\infty}^{\infty} \frac{ds}{2\pi} \exp[(a + is)N] T_n(b_{1,\dots,n}, A_n, h_{1,\dots,2^n} | s), \quad (10)$$

where $T_n(b_{1,\dots,n}, A_n, h_{1,\dots,2^n} | s)$ is the trace over the spins of a model with n levels

$$\begin{aligned} T_n(b_{1,\dots,n}, A_n, h_{1,\dots,2^n} | s) = & \int d\mathbf{S} \exp \left[\beta \sum_{i=1}^{2^n} h_i S_i - \left(a + \frac{\beta A_n}{2} + is \right) \sum_{i=1}^{2^n} S_i^2 \right], \\ & \times \exp \left[\frac{\beta}{2} L_n(S_{1,\dots,2^n}, b_{1,\dots,n}) \right] \end{aligned} \quad (11)$$

and $L_n(S_{1,\dots,n}, b_{1,\dots,n})$ encodes the hierarchical interactions of the Hamiltonian

$$L_n(S_{1,\dots,2^n}, b_{1,\dots,n}) = \sum_{p=1}^n b_p \sum_{r=1}^{2^{n-p}} \left(\sum_{i=1}^{2^p} S_{(r-1)2^p+i} \right)^2. \quad (12)$$

The shorthand notation $x_{1,\dots,S} \equiv (x_1, \dots, x_S)$ has been introduced to denote sets of variables.

The central idea consists in deriving a recursion equation between the trace of a system with n levels and the trace of a system with $n-1$ levels, but with renormalized parameters. This is achieved by making the following change of integration variables in eq. (11)

$$S_i^\pm = \frac{1}{\sqrt{2}} (S_{2i-1} \pm S_{2i}), \quad i = 1, \dots, 2^{n-1}, \quad (13)$$

which allows us to compute explicitly the Gaussian integrals over $\{S_i^-\}_{i=1,\dots,2^{n-1}}$, provided a is chosen such that $a > -\beta A_n/2$. The fulfillment of the latter condition ensures the convergence of the integrals over $\{S_i^-\}_{i=1,\dots,2^{n-1}}$. The number of degrees of freedom is reduced by one half after integrating $\{S_i^-\}_{i=1,\dots,2^{n-1}}$ out, and the variables $\{S_i^+\}_{i=1,\dots,2^{n-1}}$ enter in the definition of a function T_{n-1} , which has the same formal structure as eq. (11), but with renormalized parameters. Consequently, one can apply the above change of integration variables ℓ times in a consecutive way, obtaining the following relation between $T_n(b_{1,\dots,n}, A_n, h_{1,\dots,2^n} | s)$ in the original system and $T_{n-\ell}(b_{1,\dots,n-\ell}^{(\ell)}, A_n^{(\ell)}, h_{1,\dots,2^{n-\ell}}^{(\ell)} | s)$ in a system with $n-\ell$ levels

$$\begin{aligned} T_n(b_{1,\dots,n}, A_n, h_{1,\dots,2^n} | s) &= T_{n-\ell}(b_{1,\dots,n-\ell}^{(\ell)}, A_n^{(\ell)}, h_{1,\dots,2^{n-\ell}}^{(\ell)} | s) \\ &\times \exp \left[\frac{\beta^2}{8} \sum_{p=1}^{\ell} \frac{1}{\left(a + is + \frac{\beta}{2} A_n^{(p-1)}\right)} \sum_{r=1}^{2^{n-p}} \left(h_{2r-1}^{(p-1)} - h_{2r}^{(p-1)}\right)^2 \right] \\ &\times \exp \left[-\frac{1}{2} \sum_{p=1}^{\ell} 2^{n-p} \ln \left[\frac{1}{\pi} \left(a + is + \frac{\beta}{2} A_n^{(p-1)}\right) \right] \right], \end{aligned} \quad (14)$$

where the renormalized parameters fulfill

$$b_p^{(\ell)} = 2^\ell b_{p+\ell}, \quad (15)$$

$$A_n^{(\ell)} = A_n - \sum_{p=1}^{\ell} 2^p b_p, \quad (16)$$

$$h_i^{(\ell)} = \frac{1}{2^{\frac{\ell}{2}}} \sum_{j=1}^{2^\ell} h_{2^\ell i + 1 - j}, \quad i = 1, \dots, 2^{n-\ell}. \quad (17)$$

By setting $\ell = n$ in eq. (14), its right hand side depends on the trace $T_0(A_n^{(n)}, h_1^{(n)} | s)$ of a renormalized single-spin problem. Since T_0 is defined in terms of a Gaussian integral over a single spin variable, this object can be computed in a straightforward way, leading to an explicit expression for $T_n(b_{1,\dots,n}, A_n, h_{1,\dots,2^n} | s)$ as a function of the renormalized parameters. Substituting this explicit form of T_n in eq. (10), expressing the renormalized

parameters in terms of those of the original model through eqs. (15-17), and changing the integration variable from s to $z = (2/\beta) \left(a + is - \beta \lambda_{n+1}^{(n)} / 2 \right)$, we obtain

$$\mathcal{Z}_N = \frac{\beta}{4\pi i} \int_{\text{Re}z-i\infty}^{\text{Re}z+i\infty} dz \exp [N \Phi_N(z | h_{1,\dots,2^n})], \quad (18)$$

with

$$\begin{aligned} \Phi_N(z | h_{1,\dots,2^n}) &= \frac{1}{2} \ln \left(\frac{2\pi}{\beta} \right) + \frac{\beta}{2} \left(z + \lambda_{n+1}^{(n)} \right) - \frac{1}{2} \sum_{p=1}^n \frac{1}{2^p} \ln \left(z + \lambda_{n+1}^{(n)} - \lambda_p^{(n)} \right) \\ &- \frac{1}{2N} \ln z + \frac{\beta}{2z} \left(\frac{1}{N} \sum_{i=1}^N h_i \right)^2 \\ &+ \frac{\beta}{2N} \sum_{p=1}^n \frac{1}{2^p \left(z + \lambda_{n+1}^{(n)} - \lambda_p^{(n)} \right)} \sum_{r=1}^{2^{n-p}} \left[\sum_{i=1}^{2^{p-1}} \left(h_{(2r-1)2^{p-1}+1-i} - h_{r2^{p-1}+1-i} \right) \right]^2. \end{aligned} \quad (19)$$

All Gaussian integrals over the spin variables, involved in the derivation of eqs. (18) and (19), are convergent provided $a > -\beta A_n^{(n)}/2$. From eqs. (4) and (16), one can show that $A_n^{(\ell)} = -\lambda_{\ell+1}^{(n)}$ ($\ell = 0, \dots, n$). The condition $a > \beta \lambda_{n+1}^{(n)}/2$ is also found in the approach based on the diagonalization of \mathbf{J} [33]. We point out that eqs. (18) and (19) are completely general in the sense that they hold for a finite realisation of the random parts r_1, \dots, r_{2^n} of the local fields h_1, \dots, h_{2^n} , independently of their distribution $p(r)$.

In order to make further progress, let us assume that $\{r_i\}_{i=1,\dots,N}$ are independently drawn from eq. (1). In this case, we have checked numerically that the standard deviation of the random part of eq. (19) is of $O(1/\sqrt{N})$ for $N \gg 1$. Hence, the function $\Phi_N(z | h_{1,\dots,2^n})$ is a self-averaging quantity, *i.e.*, it converges, in the limit $N \rightarrow \infty$, to its average value

$$\Phi(z) = \frac{1}{2} \ln \left(\frac{2\pi}{\beta} \right) + \frac{\beta}{2} (z + \lambda_\infty) + \frac{\beta h^2}{2z} - g(z) + \beta \sigma^2 \frac{\partial g(z)}{\partial z}, \quad (20)$$

where $g(z)$ is written in terms of the density of eigenvalues given in eq. (5):

$$g(z) = \frac{1}{2} \int d\lambda \rho(\lambda) \ln(z + \lambda_\infty - \lambda). \quad (21)$$

It follows that the integral of eq. (18) can be solved, in the limit $N \rightarrow \infty$, through the saddle-point method, leading to the free-energy

$$f(z) = -\frac{1}{\beta} \Phi(z), \quad (22)$$

with the order-parameter z satisfying the saddle-point equation

$$\frac{\beta}{2} \left(1 - \frac{h^2}{z^2} \right) = \frac{\partial g(z)}{\partial z} - \beta \sigma^2 \frac{\partial^2 g(z)}{\partial^2 z}. \quad (23)$$

The magnetisation per spin m is obtained from the derivative of eq. (22) with respect to h

$$m = \frac{h}{z}. \quad (24)$$

The substitution of $z = h/m$ in eq. (23) yields the equation of state for this model. In the next subsection we explain how the same results are derived using the replica method.

3.2. The replica method

In the replica method, the average of the free-energy over the quenched disorder is computed using the following identity

$$\overline{\ln \mathcal{Z}_N} = \lim_{q \rightarrow 0} \frac{\partial}{\partial q} \overline{(\mathcal{Z}_N)^q}, \quad (25)$$

where $\overline{(\dots)}$ denotes the average over the distribution of the random fields. The strategy consists in calculating $\overline{(\mathcal{Z}_N)^q}$ for integer q , which corresponds to averaging the product of the partition functions of q identical copies or replicas of the system. After the thermodynamic limit is performed, q is continued analytically to real values and finally to zero. In the problem at hand, we will show that the replica symmetric (RS) *ansatz* for the saddle-point leads, in the limit $q \rightarrow 0$, to the same free-energy as that computed with the recursive method.

Substituting eq. (2) in eq. (8) and performing the average of the replicated partition function $(\mathcal{Z}_N)^q$ over the random variables $\{r_i\}_{i=1,\dots,N}$, we obtain

$$\begin{aligned} \overline{(\mathcal{Z}_N)^q} &= \int \left(\prod_{\alpha=1}^q d\mathbf{S}^\alpha \right) \int \left(\prod_{\alpha=1}^q \frac{ds_\alpha}{2\pi} \right) \exp \left[N \sum_{\alpha=1}^q (a + is_\alpha) - \sum_{\alpha=1}^q (\mathbf{S}^\alpha)^T \cdot \mathbf{V}(s_\alpha) \mathbf{S}^\alpha \right] \\ &\times \exp \left[\beta \sum_{\alpha=1}^q \mathbf{u}^T \cdot \mathbf{S}^\alpha + \frac{\beta^2 \sigma^2}{2} \sum_{\alpha,\beta=1}^q (\mathbf{S}^\alpha)^T \cdot \mathbf{S}^\beta \right], \end{aligned} \quad (26)$$

where $(\mathbf{S}^\alpha)^T = (S_1^\alpha, \dots, S_N^\alpha)$ is the global state vector in a given replica α , while $\mathbf{u}^T = (h, \dots, h)$ is the N -dimensional vector including the uniform part of the external fields. The matrix $\mathbf{V}(s_\alpha)$ is defined as follows

$$\mathbf{V}(s_\alpha) = (a + is_\alpha) \mathbf{I} - \frac{\beta}{2} \mathbf{J}, \quad (27)$$

with \mathbf{I} denoting the $N \times N$ identity matrix.

Let us define the set of normalised eigenvectors and eigenvalues of $\mathbf{V}(s_\alpha)$ by $\{\phi_k\}_{k=1,\dots,N}$ and $\{v_k(s_\alpha)\}_{k=1,\dots,N}$, respectively. The eigenvectors are independent of the replica index α , since they are the same as the eigenvectors of \mathbf{J} . The insertion of the completeness relation for $\{\phi_k\}_{k=1,\dots,N}$ in each term of eq. (26) yields the following expression

$$\begin{aligned} \overline{(\mathcal{Z}_N)^q} &= \int \left(\prod_{\alpha=1}^q \prod_{k=1}^N dP_k^\alpha \right) \int \left(\prod_{\alpha=1}^q \frac{ds_\alpha}{2\pi} \right) \exp \left[N \sum_{\alpha=1}^q (a + is_\alpha) \right] \\ &\times \prod_{k=1}^N \exp \left[-\frac{1}{2} \sum_{\alpha,\beta=1}^q P_k^\alpha A_k^{\alpha\beta}(s_\alpha) P_k^\beta + \beta y_k \sum_{\alpha=1}^q P_k^\alpha \right], \end{aligned} \quad (28)$$

where the scalar projections $P_k^\alpha = (\mathbf{S}^\alpha)^T \cdot \boldsymbol{\phi}_k$ and $y_k = \mathbf{u}^T \cdot \boldsymbol{\phi}_k$ onto the eigenvectors $\{\boldsymbol{\phi}_k\}_{k=1,\dots,N}$ have been introduced. The elements of the q -dimensional matrix $\mathbf{A}_k(s_{1,\dots,q})$ in the replica space are given by

$$A_k^{\alpha\beta}(s_\alpha) = 2v_k(s_\alpha)\delta_{\alpha\beta} - \beta^2\sigma^2. \quad (29)$$

As usual in the replica method, the original model with quenched random fields has been converted in a pure system composed of replicated variables P_k^1, \dots, P_k^q that interact through the off-diagonal elements of $\mathbf{A}_k(s_{1,\dots,q})$, as can be noted from eq. (28). By assuming $a > \beta\lambda_{n+1}^{(n)}/2 + \beta^2\sigma^2/2$, we can integrate over P_k^α and derive

$$\overline{(\mathcal{Z}_N)^q} = (2\pi)^{\frac{Nq}{2}} \int \left(\prod_{\alpha=1}^q \frac{ds_\alpha}{2\pi} \right) \exp [NW(s_{1,\dots,q})], \quad (30)$$

where we have introduced the action in the replica space

$$W(s_{1,\dots,q}) = \sum_{\alpha=1}^q (a + is_\alpha) - \frac{1}{2N} \sum_{k=1}^N \ln \det \mathbf{A}_k(s_{1,\dots,q}) + \frac{\beta^2}{2N} \sum_{k=1}^N y_k^2 \sum_{\alpha,\beta=1}^q [\mathbf{A}_k^{-1}(s_{1,\dots,q})]_{\alpha\beta}. \quad (31)$$

In order to obtain the free-energy we make the replica symmetric *ansatz*, i.e., we assume that $s_\alpha = s$ for $\alpha = 1, \dots, q$, from which one can show that

$$\begin{aligned} \det \mathbf{A}_k(s) &= [2v_k(s)]^q \left[1 - \frac{q\beta^2\sigma^2}{2v_k(s)} \right], \\ [\mathbf{A}_k^{-1}(s)]_{\alpha\beta} &= \frac{1}{2v_k(s)} \delta_{\alpha\beta} + \frac{\beta^2\sigma^2}{2v_k(s) [2v_k(s) - q\beta^2\sigma^2]}. \end{aligned} \quad (32)$$

The correctness of the RS assumption will be justified through a local stability analysis in the following section. The insertion of the above two equations in eq. (30) reads

$$\begin{aligned} \overline{(\mathcal{Z}_N)^q} &= (2\pi)^{\frac{Nq}{2}} \int \frac{ds}{2\pi} \exp \left[Nq(a + is) + \frac{\beta^2 q}{2} \sum_{k=1}^N \frac{y_k^2}{2v_k(s) - q\beta^2\sigma^2} - \frac{q}{2} \sum_{k=1}^N \ln [2v_k(s)] \right] \\ &\times \exp \left[-\frac{1}{2} \sum_{k=1}^N \ln \left(1 - \frac{q\beta^2\sigma^2}{2v_k(s)} \right) \right]. \end{aligned} \quad (33)$$

By noting that $\mathbf{u} = \sqrt{N}h\boldsymbol{\phi}_N$, with $\boldsymbol{\phi}_N = N^{-1/2}(1, \dots, 1)$ representing the uniform eigenvector of \mathbf{J} corresponding to the largest eigenvalue $\lambda_{n+1}^{(n)}$, we have that $y_k = \sqrt{N}h\delta_{N,k}$, which follows from the orthogonality among the eigenvectors. This implies that only the term with $k = N$ survives in the contribution involving $\{y_k^2\}_{k=1,\dots,N}$. The last step consists in inserting the explicit form of $\{v_k(s)\}_{k=1,\dots,N}$, given in terms of the eigenvalues $\{\lambda_p^{(n)}\}_{p=1,\dots,n+1}$ of \mathbf{J} , with the corresponding degeneracy factors as defined just below eq. (4), and to make the change of integration variable $z = (2/\beta) \left(a + is - \beta\lambda_{n+1}^{(n)}/2 \right)$, to obtain

$$\overline{(\mathcal{Z}_N)^q} = \frac{\beta}{4\pi i} \int_{\text{Re}z - i\infty}^{\text{Re}z + i\infty} dz \exp [N\Phi_q^N(z)]. \quad (34)$$

In the limit $N \rightarrow \infty$, the sums $\sum_{k=1}^N(\dots)$ in eq. (33) may be replaced by averages with the density of eigenvalues $\rho(\lambda)$, and the function $\Phi_q^N(z)$ converges to the following well-defined limit

$$\Phi_q(z) = \frac{q}{2} \ln \left(\frac{2\pi}{\beta} \right) + \frac{q\beta}{2} (z + \lambda_\infty) + \frac{q\beta h^2}{2(z - q\beta\sigma^2)} - qg(z) - g_q(z), \quad (35)$$

where $g(z)$ is defined by eq. (21), and $g_q(z)$ reads

$$g_q(z) = \frac{1}{2} \int d\lambda \rho(\lambda) \ln \left[1 - \frac{q\beta\sigma^2}{(z + \lambda_\infty - \lambda)} \right]. \quad (36)$$

The integral in eq. (34) is computed, in the limit $N \rightarrow \infty$, through the saddle-point method, and the free-energy reads

$$f(z) = -\frac{1}{N\beta} \overline{\ln \mathcal{Z}_N} = -\frac{1}{\beta} \lim_{q \rightarrow 0} \frac{\partial \Phi_q(z)}{\partial q} = -\frac{1}{\beta} \Phi(z), \quad (37)$$

where $\Phi(z)$ is defined by eq. (20) and z fulfills the saddle-point equation (23). The magnetization is computed in the replica method as follows

$$m = \frac{1}{N\beta} \frac{\partial}{\partial h} \overline{\ln \mathcal{Z}_N} = \frac{1}{\beta} \frac{\partial}{\partial h} \lim_{q \rightarrow 0} \frac{\partial \Phi_q(z)}{\partial q} = \frac{h}{z}. \quad (38)$$

Equations (37) and (38) are the same as those derived in the previous section by means of the recursive approach.

4. Phase diagram and the stability of the RS saddle-point

In this section we discuss the properties of the equation of state and the local stability of the RS assumption of the previous section. These studies allow us to provide a rather complete characterisation of the phase diagram.

4.1. Phase diagram

The thermodynamics of the model is governed by two parameters: β and σ . In order to study the phase diagram, we express the free energy, given in eq. (37), in terms of the magnetisation $m = h/z$

$$f(m, h) = f_{\text{pure}}(m, h) - \sigma^2 g' \left(\frac{h}{m} \right), \quad (39)$$

where $g'(z) = \frac{\partial g(z)}{\partial z}$, and $f_{\text{pure}}(m, h)$ is the free-energy $f(m, h)$ calculated at $\sigma = 0$:

$$f_{\text{pure}}(m, h) = -\frac{1}{2\beta} \ln \left(\frac{2\pi}{\beta} \right) - \frac{1}{2} \left(\frac{h}{m} + \lambda_\infty \right) - \frac{hm}{2} + \frac{1}{\beta} g \left(\frac{h}{m} \right). \quad (40)$$

The equation of state is obtained by writing the saddle-point equation (23) in terms of m and h

$$1 - m^2 = \frac{2}{\beta} g' \left(\frac{h}{m} \right) - 2\sigma^2 g'' \left(\frac{h}{m} \right). \quad (41)$$

In the ferromagnetic phase, where $|m| > 0$, we can safely send h to zero in eq. (41), obtaining an equation in terms of $g'(0)$ and $g''(0)$, which depends on the magnetisation

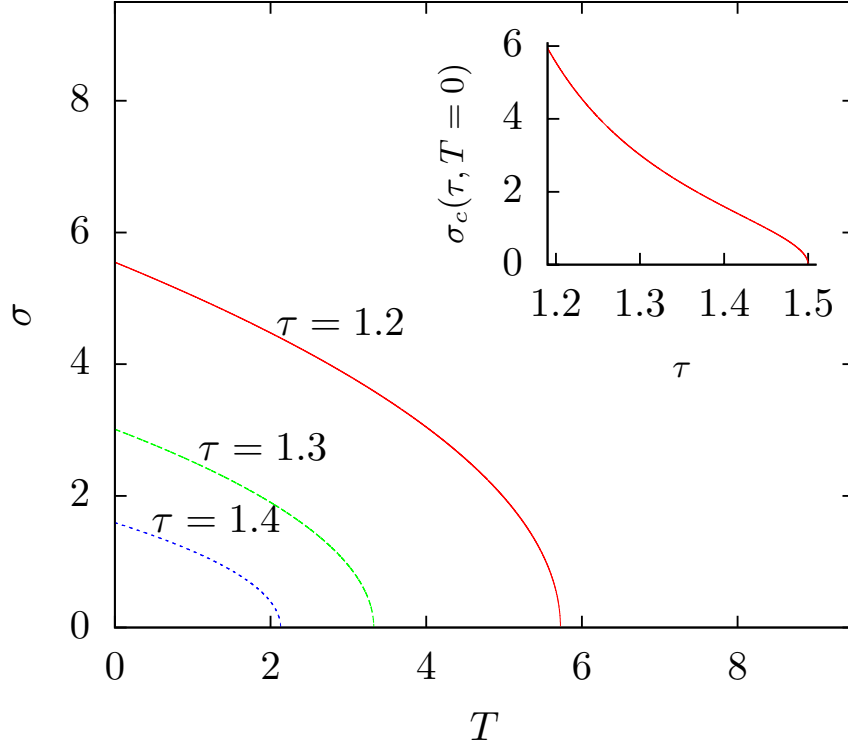


Figure 2. Phase diagram of the hierarchical spherical model with random fields for different values of τ . The system is in a paramagnetic or in a ferromagnetic phase depending if the parameters are chosen above or below the critical line, respectively. The inset displays the critical standard deviation of the random fields as function of τ at zero temperature.

m . The resulting equation is satisfied at the critical values β_c and σ_c , by definition, if $m = 0$. Thus, the critical line separating the ferromagnetic from the paramagnetic phase on the $\sigma - \beta$ reads

$$\frac{\beta_c}{2} = g'(0) - \sigma_c^2 \beta_c g''(0). \quad (42)$$

Equation (42) is a self-consistent equation that gives a value of the critical variance σ_c^2 for each value of the temperature $T_c = 1/\beta_c$, and vice versa. However, the existence of a solution of this equation depends on the value of τ , which governs the analyticity properties of the functions $g'(z)$ and $g''(z)$ around $z = 0$. In the appendix, we show that $g'(0)$ and $g''(0)$ are finite provided $\tau < 2$ and $\tau < 3/2$, respectively. This means that, for $\tau < 3/2$, a solution $\sigma_c^2(T) \geq 0$ exists, and there is a critical line that connects the zero temperature critical point $\sigma_c^2(0)$ with the finite temperature critical point $\beta_c(\sigma^2 = 0) = 2g'(0)$ of the pure model. For $\sigma^2 > 0$, the critical line shrinks to the T axis when $\tau \rightarrow 3/2$, so that no phase transition is present for $\tau \geq 3/2$. For $\sigma^2 = 0$, the model has a finite critical temperature which goes to zero when $\tau \rightarrow 2$.

The phase diagram is reported in Fig. 2. Below the critical line, the system is in a ferromagnetic phase, where the magnetisation is different from zero. Although

we have $m = 0$ above the critical line, an intermediate spin-glass phase can not be excluded. It has been shown rigorously in [8, 9] that certain random field models undergo a transition between a paramagnetic and a ferromagnetic phase, without the appearance of any intermediate spin-glass phase. Despite these results, we cannot exclude *a priori* an intermediate spin-glass phase, since the present model does not belong to the class of systems studied in [8, 9]. In the following we will show that the replica symmetric solution is locally stable. This result, combined with the fact that we obtain the same free-energy through the recursive and the replica method, strongly supports the absence of a spin-glass phase.

4.2. Stability of the RS ansatz

In this section we study the stability properties of the matrix that governs the small fluctuations around the RS saddle-point. This is important in order to see if there is the possibility of a spin-glass phase with replica symmetry breaking. Let us consider the action defined by eq. (31). The small fluctuations around the RS saddle-point are controlled by the eigenvalues of the stability matrix [40]

$$M_{\alpha\beta} = \left. \frac{\partial^2 W}{\partial s_\alpha \partial s_\beta} \right|_{RS}, \quad (43)$$

where $(\dots)|_{RS}$ means that the second derivatives are calculated at the RS saddle-point, characterised by $s_\alpha = s$ for $\alpha = 1, \dots, q$.

Equation (29), combined with the explicit form for the eigenvalues $v_k(s_\alpha)$, allows us to show that $\partial (\mathbf{A}_k)_{\alpha\beta} / \partial s_\gamma = 2i\delta_{\alpha\beta}\delta_{\alpha\gamma}$. By using this result and the following general relations

$$\frac{\partial (\mathbf{A}_k^{-1})_{\alpha\beta}}{\partial (\mathbf{A}_k)_{\gamma\delta}} = -(\mathbf{A}_k^{-1})_{\alpha\gamma} (\mathbf{A}_k^{-1})_{\delta\beta}, \quad \frac{\partial}{\partial (\mathbf{A}_k)_{\alpha\beta}} \ln \det \mathbf{A}_k = (\mathbf{A}_k^{-1})_{\beta\alpha}, \quad (44)$$

one can derive the expression for the second derivative

$$\begin{aligned} \frac{\partial^2 W}{\partial s_\gamma \partial s_\omega} = & -\frac{2}{N} \sum_{k=1}^N (\mathbf{A}_k^{-1})_{\gamma\omega} (\mathbf{A}_k^{-1})_{\omega\gamma} - \frac{2\beta^2}{N} \sum_{k=1}^N y_k^2 \sum_{\alpha,\beta=1}^q (\mathbf{A}_k^{-1})_{\alpha\omega} (\mathbf{A}_k^{-1})_{\omega\gamma} (\mathbf{A}_k^{-1})_{\gamma\beta} \\ & - \frac{2}{N} \sum_{k=1}^N \sum_{\alpha,\beta=1}^q (\mathbf{A}_k^{-1})_{\alpha\gamma} (\mathbf{A}_k^{-1})_{\gamma\omega} (\mathbf{A}_k^{-1})_{\omega\beta}. \end{aligned} \quad (45)$$

We need to compute the stability matrix at the RS solution. In this case, the inverse \mathbf{A}_k^{-1} is given by eq. (32), and the elements of the stability matrix at the RS saddle-point assume the form $M_{\alpha\beta} = M_1\delta_{\alpha\beta} + M_2$, where

$$M_1 = -\frac{1}{2N} \sum_{k=1}^N A_D^{(k)} \left[2\beta^2 y_k^2 \left(A_D^{(k)} + qA_F^{(k)} \right)^2 + A_D^{(k)} + 2A_F^{(k)} \right], \quad (46)$$

$$M_2 = -\frac{1}{2N} \sum_{k=1}^N A_F^{(k)} \left[2\beta^2 y_k^2 \left(A_D^{(k)} + qA_F^{(k)} \right)^2 + \left(A_F^{(k)} \right)^2 \right], \quad (47)$$

with the coefficients

$$A_D^{(k)} = \frac{1}{2v_k(s)}, \quad A_F^{(k)} = \frac{\beta^2 \sigma^2}{2v_k(s)(2v_k(s) - q\beta^2 \sigma^2)}, \quad (48)$$

which are the diagonal and the off-diagonal parts of $(\mathbf{A}^{-1})_{\alpha\beta}$, as can be seen from eq. (32).

The eigenvalues of the stability matrix are given by

$$\lambda_1 = M_1 + qM_2, \quad \lambda_2 = M_1. \quad (49)$$

The degeneracy of λ_1 is 1 while the degeneracy of λ_2 is $q - 1$. We want to compute, in the limit $q \rightarrow 0$, the eigenvalues λ_1 and λ_2 at the solution z of the saddle-point equation (23). In the limit $h \rightarrow 0$ of vanishing external field, z tends to a value z_0 and one can show that $z_0 = 0$ in the ferromagnetic phase, while $z_0 > 0$ in the paramagnetic phase (see the next section). Thus, in the absence of external field, the eigenvalues $v_k(s)$, expressed as a function of z_0 , are given by $v_k(z_0) = (\beta/2)(z_0 + \lambda_{n+1}^{(n)} - \lambda_k^{(n)}) \geq 0$, which implies that $A_D^{(k)}$ and $A_F^{(k)}$ are positive quantities for $q \rightarrow 0$. As a consequence, the functions M_1 and M_2 , defined by eqs. (46) and (47), are negative in the limit $q \rightarrow 0$. Thus, the eigenvalues λ_1 and λ_2 of the stability matrix are negative in the whole phase diagram, implying the local stability of the RS solution.

5. Critical exponents

In order to compute the critical exponents, we need to study eq. (23) close to the critical line. Let us define $K = \beta/2$. As $h \rightarrow 0^+$, the l.h.s. of eq. (23), $L_h(z) = K(1 - h^2/z^2)$, is more and more close to the constant value K , while the r.h.s. $R_h(z) = g'(z) - 2K\sigma^2 g''(z)$ is a smoothly decaying function. Let us define $z_0 = \lim_{h \rightarrow 0} z^*(h)$, where $z^*(h)$ is the intersection point between $L_h(z)$ and $R_h(z)$. The function $R_{h \rightarrow 0^+}(z = 0)$ determines whether or not $m > 0$: for $R_{h \rightarrow 0^+}(z = 0) > K$, z_0 is finite and positive, and thus $m = 0$, while for $R_{h \rightarrow 0^+}(z = 0) < K$, z_0 is zero, and thus $m > 0$. We are going to use eq. (23) to compute the critical exponents.

In this model the critical point may be approached following different directions in the (T, σ) phase diagram. We limit ourselves to discuss the situations when we vary σ at a fixed value of T , and vice versa. The critical behaviour does not depend on this choice. Since the critical exponents depend crucially on the behaviour of the solution z_0 of eq. (23) close to the critical line, we first study such behavior coming from the paramagnetic phase and approaching the critical line.

Let us first study the case when T is fixed and we vary σ^2 . The critical value $\sigma_c^2(T)$ reads

$$\sigma_c^2(T) = \frac{g'(0) - K}{2Kg''(0)}. \quad (50)$$

In the paramagnetic phase, we have that $z_0 > 0$ and the limit $h \rightarrow 0$ can be safely taken in eq. (23) to obtain

$$K = g'(z_0) - 2K\sigma^2 g''(z_0). \quad (51)$$

Close to the critical line, we have to study the behaviour of $g'(z_0)$ and $g''(z_0)$ for small z_0 . We focus on the region where $\tau < 3/2$, which is the relevant one for the random field model, namely for $\sigma^2 > 0$. The behaviour of z_0 depends crucially on the value of τ . In fact, by using the results presented in the appendix, we can show that for $\tau < 4/3$ we have

$$z_0 \sim (\sigma^2 - \sigma_c^2(T)), \quad (52)$$

while for $\tau > 4/3$ the equation of state becomes

$$0 = (\sigma^2 - \sigma_c^2(T))g''(0) + \sigma^2 C_1 z_0^{\frac{3-2\tau}{\tau-1}}, \quad (53)$$

with C_1 denoting an unimportant constant. For $\tau > 4/3$, $g'''(0)$ does not exist, but $g''(z)$ is given by $g''(0) + C_1 z^{\frac{3-2\tau}{\tau-1}}$ (at the leading order in $z \ll 1$), because its derivative $g'''(z)$ is proportional to $z^{\frac{4-3\tau}{\tau-1}}$. For $\sigma \rightarrow \sigma_c(T)$, this means that

$$z_0 \sim \begin{cases} [\sigma - \sigma_c(T)] & \text{if } \tau < \frac{4}{3}, \\ [\sigma - \sigma_c(T)]^{\frac{\tau-1}{3-2\tau}} & \text{if } \frac{4}{3} < \tau < \frac{3}{2}. \end{cases} \quad (54)$$

By repeating exactly the same analysis for fixed σ^2 , one can show that, for $K \rightarrow K_c(\sigma)$, z_0 behaves as follows

$$z_0 \sim \begin{cases} [K - K_c(\sigma)] & \text{if } \tau < \frac{4}{3}, \\ [K - K_c(\sigma)]^{\frac{\tau-1}{3-2\tau}} & \text{if } \frac{4}{3} < \tau < \frac{3}{2}. \end{cases} \quad (55)$$

For the pure model, where $\sigma^2 = 0$, the r.h.s. of eq. (23) is well defined in a larger domain $\tau < 2$ and, in a similar way, we obtain, for $K \rightarrow K_c(\sigma = 0)$, the following result

$$z_0 \sim \begin{cases} [K - K_c(\sigma = 0)] & \text{if } \tau < \frac{3}{2}, \\ [K - K_c(\sigma = 0)]^{\frac{\tau-1}{2-\tau}} & \text{if } \tau > \frac{3}{2}. \end{cases} \quad (56)$$

The critical behaviour of the pure model is, in general, different with respect to the case $\sigma^2 > 0$. The only critical exponent that has the same value in the random field model and in the pure model is β , as we will show in the sequel. We will further see that eqs. (54) and (55) imply that the critical exponents for $\sigma^2 > 0$ do not depend on the direction that the critical line is crossed. Moreover, the critical exponents for $T > 0$ present the same values as in the regime $T = 0$. This is a well established property of random field systems, whose reasons can be found using renormalization group arguments [41].

5.1. Calculation of β

The critical exponent β is defined, in the limit $h \rightarrow 0^+$, by the vanishing of the magnetisation as the critical line is approached from the ferromagnetic phase [33]. Let us consider the case where T is held at a fixed value and σ^2 is the control parameter. Using eq. (50), equation (23) reduces, in the ferromagnetic phase, to the form

$$m^2 = 2g''(0)[\sigma_c^2(T) - \sigma^2], \quad (57)$$

from which we obtain that, for $T \geq 0$, the magnetisation vanishes as

$$m \sim \sqrt{\sigma_c^2(T) - \sigma^2}. \quad (58)$$

In the above derivation, we have used that $g''(0)$ is finite for $\tau < 3/2$, as can be seen from the results shown in the appendix. We can also study the behaviour of m as a function of T for fixed σ^2 . Using the fact that

$$\beta_c(\sigma) = \frac{2g'(0)}{1 + 2\sigma^2 g''(0)} , \quad (59)$$

eq. (23) reduces, in the ferromagnetic phase, to the form

$$m^2 = \frac{1 + 2\sigma^2 g''(0)}{\beta} [\beta - \beta_c(\sigma)] , \quad (60)$$

from which we have that, for $\sigma^2 \geq 0$, the magnetisation vanishes according to

$$m \sim \sqrt{\beta - \beta_c(\sigma)} , \quad (61)$$

where we have used again that $g''(0)$ is finite for $\tau < 3/2$. From equations (58) and (61), we obtain the critical exponent $\beta = 1/2$. This is also the value of β in the pure model, as can be noted from eq. (61).

5.2. Calculation of γ

The critical exponent γ is defined from the divergence of the zero-field susceptibility, defined as $\chi = \lim_{h \rightarrow 0^+} \partial m / \partial h$, as we approach the critical line from the paramagnetic phase [33]. The magnetisation is given by $m = h/z$, so that

$$\chi = \frac{1}{z_0} - \lim_{h \rightarrow 0^+} \frac{h}{z^2} \frac{\partial z}{\partial h} , \quad (62)$$

where z here is the short hand notation for $z^*(h)$, namely the solution of eq. (23) for $h > 0$. In the paramagnetic phase, z_0 has a finite value that vanishes as we approach the critical line. Using the equation of state, it can be shown that

$$\frac{h}{z^2} \frac{\partial z}{\partial h} = \frac{2Kh^2}{2Kh^2/z^3 - g''(z) - z^2\beta^2 g'''(z)} . \quad (63)$$

From the results of the appendix, the above equation vanishes for $h \rightarrow 0^+$ within the paramagnetic phase, which implies that $\chi = 1/z_0$. Thus, by considering the behaviour of z_0 close to the critical line, we derive the following results for the critical exponent γ

$$\gamma = \begin{cases} 1 & \text{if } \tau < \frac{4}{3} \\ \frac{\tau-1}{3-2\tau} & \text{if } \frac{4}{3} < \tau < \frac{3}{2} \end{cases} . \quad (64)$$

Equation (64) holds whatever direction we choose to cross the critical line, including the case of $T = 0$. A similar analysis can be done for the pure model. Using eq. (56), we get the result obtained in reference [31]:

$$\gamma = \begin{cases} 1 & \text{if } \tau < \frac{3}{2} \\ \frac{\tau-1}{2-\tau} & \text{if } \tau > \frac{3}{2} \end{cases} . \quad (65)$$

5.3. Calculation of α

The critical exponent α is defined, for $h = 0$, from the divergence of the specific heat at the critical line [33]. If $u(m, h, T)$ denotes the energy density as a function of T at fixed σ^2 , then α can be computed from

$$u(m, 0, T_c(\sigma) + \varepsilon/2) - u(m, 0, T_c(\sigma) - \varepsilon/2) \sim |\varepsilon|^{1-\alpha}, \quad (66)$$

with $|\varepsilon| \rightarrow 0$. In the case where the energy density is a function of σ for fixed T , the exponent α is defined in an analogous way. The function u is calculated above and below the critical line from the free energy as $u = \partial(\beta f)/\partial\beta$. Dropping the dependency on T or σ , we use eqs. (39) and (40) to obtain

$$u(m, h) = u_{\text{pure}}(m, h) - \sigma^2 g' \left(\frac{h}{m} \right) - \beta \sigma^2 g'' \left(\frac{h}{m} \right) \frac{\partial z_0}{\partial \beta}, \quad (67)$$

where $u_{\text{pure}} = \partial(\beta f_{\text{pure}})/\partial\beta$. Note that u_{pure} is not the energy density of the pure model, since the saddle-point value of m depends on σ^2 . Equation (67) allows us to compute u above and below the critical line, employing the results from the appendix combined with eqs. (54) and (55). As a result, we obtain the values of the critical exponent α :

$$\alpha = \begin{cases} 0 & \text{if } \tau < \frac{4}{3} \\ \frac{4-3\tau}{3-2\tau} & \text{if } \frac{4}{3} < \tau < \frac{3}{2} \end{cases}. \quad (68)$$

This result holds whatever direction we cross the critical line, including the case of $T = 0$. A similar analysis, together with eq. (56), leads to the following result for the pure model

$$\alpha = \begin{cases} 0 & \text{if } \tau < \frac{3}{2} \\ \frac{3-2\tau}{2-\tau} & \text{if } \tau > \frac{3}{2} \end{cases}. \quad (69)$$

5.4. Calculation of δ

The critical exponent δ is obtained from the vanishing of the magnetisation as a function of h at the critical line according to $m \sim h^{1/\delta}$ [33]. From eqs. (41) and (42), we obtain that $h^2 \sim z_0^3$ if $\tau < 4/3$, while $h^2 \sim z_0^{1/(\tau-1)}$ if $\tau > 4/3$. Thus, eqs. (54) and (55) lead to the results

$$\delta = \begin{cases} 3 & \text{if } \tau < \frac{4}{3} \\ \frac{1}{3-2\tau} & \text{if } \frac{4}{3} < \tau < \frac{3}{2} \end{cases}, \quad (70)$$

whatever direction we choose to cross the critical line. For the pure model, a similar analysis combined with eq. (56) yields

$$\delta = \begin{cases} 3 & \text{if } \tau < \frac{3}{2} \\ \frac{\tau}{2-\tau} & \text{if } \tau > \frac{3}{2} \end{cases}. \quad (71)$$

6. Considerations on the critical exponents

From the results presented above, it is clear that $\tau \in (1, 3/2)$ is the interesting interval of τ . For $\tau > 3/2$, no phase transition occurs when $\sigma^2 > 0$. The threshold $\tau = 3/2$ is called the lower critical value. The value $\tau = 4/3$, referred to as the upper critical value, plays a central role: for $\tau \in (1, 4/3)$, the mean-field theory is valid and the critical exponents assume their classical values, whereas for $\tau \in (4/3, 3/2)$ the critical exponents are non-trivial, in the sense that they are, in general, functions of τ . In the pure model, the upper and lower critical values are given, respectively, by $\tau = 3/2$ and $\tau = 2$. We remark that the same lower and upper values of τ have been found in the hierarchical model with Ising spins in the pure case [16] as well as in the presence of random fields [27, 42]. This situation is different from the D -dimensional short-range counterpart, where the spherical model and the model with Ising spins have different lower critical dimensions, given by $D = 4$ and $D = 3$, respectively [1, 34].

Differently from the spherical model, it is commonly not possible to derive, within the non-mean-field region, analytical expressions for the critical exponents in the Ising counterpart of the present model. An exception is the exponent δ . Let us define $\delta_{Pure}(\tau)$ and $\delta_{RF}(\tau)$ as the exponents for the pure and the random field model, respectively. In the hierarchical model with Ising spins [16, 27], the exponents $\delta_{RF}(\tau)$ and $\delta_{Pure}(\tau)$ have exactly the same values as presented in eqs. (70) and (71), respectively. Thus, one can conclude that the relation $\delta_{RF}(2 - 1/\tau) = \delta_{Pure}(\tau)$ is fulfilled in the respective non-mean-field regimes of hierarchical models with spherical as well as with Ising spins.

It is a natural question to ask if a similar mapping between the critical properties of the pure and the random field case holds for the other critical exponents. In the hierarchical model with Ising spins, it has been shown, using perturbation theory, that the relation $\gamma_{RF}(2 - 1/\tau) = \gamma_{Pure}(\tau)$ holds near the corresponding upper critical values of τ , but fails in the non-mean-field region [27]. In the spherical model, instead, this is not the case. From the results of the previous section, we see that this mapping is satisfied for all critical exponents, for any value of τ in the non-mean field sector.

It has been suggested that the critical properties of one-dimensional long-range models and D -dimensional short-range ones can be connected through the relation [18, 19, 25, 26, 29]

$$D = \frac{2}{\tau - 1}, \quad (72)$$

which gives the equivalent dimension D of the short-range model with the same critical behaviour as the one-dimensional long-range model parametrised by the interaction potential $J(r) \sim r^{-\tau}$. For models with Ising spins, such relation breaks down in the non-mean-field region [27, 29]. In contrast, for models with spherical spins, one can substitute τ in terms of D on the critical exponents obtained in the previous section: the resulting expressions are the same as those derived in references [31] and [34], showing that in this case there is an exact mapping between the critical behaviour of the hierarchical model and that of D -dimensional short-range systems. A possible reason for this can be

found in reference [29], where it has been shown that, under a certain super-universality hypothesis, a different relation between D and τ can be derived, which should improve eq. (72) and extend its validity to the non-mean-field regime. This relation involves the critical exponent $\eta_{SR}(D)$, such that eq. (72) is recovered for $\eta_{SR} = 0$. Since an explicit computation in the D -dimensional spherical model leads to $\eta_{SR} = 0$, equation (72) holds exactly in this case.

7. Conclusion

We have studied the equilibrium properties of a spherical version of the Dyson hierarchical model in the presence of random fields using two independent methods: a recursive computation of the partition function, based on a renormalization-like transformation, and the standard replica approach. Both methods give exactly the same free-energy and the same equation of state, from which it follows that the model undergoes a paramagnetic-ferromagnetic phase transition on the (σ^2, T) plane, with σ^2 denoting the variance of the random fields and T the temperature. By tuning a parameter τ , responsible for controlling the power-law decaying interactions, the hierarchical model interpolates smoothly between a mean-field and a non-mean-field regime. We have computed analytically the critical exponents in both regimes and their values do not depend on the direction that the critical line is crossed on the phase diagram.

Two interesting results emerge from the calculation of the critical exponents. First, there is an exact mapping between the critical behaviour of the pure model and that of the random field model. In fact, we have shown that, contrary to the Ising version of the present model [27], the relation $y_{RF}(2 - 1/\tau) = y_{Pure}(\tau)$ holds in the whole non-mean-field sector, where y denotes one of the critical exponents considered here. Such relation, which has been proposed in reference [27], plays the role of the dimensional reduction rule for one-dimensional long-range systems. Second, there is an exact mapping, given by eq. (72), between the critical properties of the spherical hierarchical model in the presence of random fields and the corresponding D -dimensional model with short-range interactions. This conclusion follows from the comparison of our results for the critical exponents with those of references [31, 34]. In contrast to the Ising version of the hierarchical model, eq. (72) is valid here for any value of τ .

Finally, from the local stability analysis of the replica symmetric solution we have shown that the model does not display a spin-glass phase. Although the emergence of spin-glass states through a discontinuous transition can not be definitely excluded, the absence of a spin-glass phase in this model is also reinforced by the fact that the free-energy obtained from the recursive approach is precisely the same as the replica symmetric free-energy. This result extends those of references [8, 9], where it has been shown rigorously that random field systems composed of Ising spins [8] or defined in terms of a scalar field theory [9] do not exhibit a spin-glass phase.

The present paper opens some interesting perspectives of future works. Since

spherical models are recovered as the $m \rightarrow \infty$ limit of $O(m)$ vectorial models in the pure case [43], it would be interesting to investigate perturbatively how eq. (72) is modified in vectorial models with m very large, but finite. Such study could lead to additional insights on the mechanism at work in the breakdown of eq. (72), which generally occurs in low dimensional systems. We point out that the connection between $O(m)$ vectorial models and spherical models holds in the pure case, but it is not obvious in systems with quenched disorder. Thus, our results on the critical exponents of the random field model and on the absence of a spin-glass phase may not trivially hold for the hierarchical model with vectorial spins in the limit of a large number of components. Another interesting perspective is the extension of our work to hierarchical spherical models with random couplings. We leave this for future work.

Acknowledgments

The authors thank Giorgio Parisi and Federico Ricci-Tersenghi for fruitful discussions and interesting suggestions. P.U. thanks the Physics Department of the University of Rome “La Sapienza” where part of this work has been developed. The research leading to these results has received funding from the European Research Council (ERC) grant agreement No. 247328 (CriPheRaSy project). F.L.M acknowledges the support from the Brazilian agency CAPES through the program Science Without Borders. P.U. acknowledges the support from the ERC grant NPRGGLASS.

Appendix A. Some useful results to calculate the critical exponents

In this appendix we study the function $g'(z)$ and its derivatives for small values of z , which are important in the evaluation of the critical exponents. Moreover, we compute $g'(0)$ and $g''(0)$, which are needed in the computation of the critical line, see eq. (42).

The function $g(z)$ is defined in eq. (21), and its derivative $g'(z)$ reads

$$g'(z) = \frac{1}{2} \int d\lambda \frac{\rho(\lambda)}{z + \lambda_\infty - \lambda} = \frac{1}{2} \lim_{n \rightarrow \infty} \sum_{p=1}^n \frac{2^{-p}}{z + \lambda_\infty - \lambda_p}. \quad (\text{A.1})$$

We note that $\lambda_\infty - \lambda_p$ is a positive quantity. This means that the function $g'(z)$ contains a sequence of simple poles on the negative part of the real z axis and the point $z = 0$ is an accumulation point for such poles. In what follows we assume $z \geq 0$, because we know from the equation of state that this is the physical relevant case. For $z > 0$, the series in eq. (A.1) is always convergent because the ratio between the $p + 1$ -th and the p -th term is smaller than one. For $z = 0$, the series converges to the value

$$g'(0) = \frac{1}{2^\tau(\lambda_\infty - \lambda_1)} \left[\frac{1}{1 - 2^{-(2-\tau)}} - 1 \right], \quad (\text{A.2})$$

provided $\tau < 2$.

We want to understand the behaviour of the above series for $\tau > 2$, in the regime of small and positive z . Let us define $\pi(z)$ as the value of p that satisfies the following equation

$$\frac{z}{\lambda_\infty - \lambda_{\pi(z)}} = 1. \quad (\text{A.3})$$

Defining $\tilde{p}(z) = \lfloor \pi(z) \rfloor$, where $\lfloor x \rfloor$ denotes the largest integer not greater than x , the behaviour of \tilde{p} , for $z \rightarrow 0$, is such that $2^{-(\tau-1)\tilde{p}} \sim z$. Thus we obtain

$$\sum_{p=1}^{\infty} \frac{2^{-p}}{z + \lambda_\infty - \lambda_p} = \sum_{p=1}^{\tilde{p}} \frac{2^{-p}}{z + \lambda_\infty - \lambda_p} + \sum_{p=\tilde{p}+1}^{\infty} \frac{2^{-p}}{z + \lambda_\infty - \lambda_p}. \quad (\text{A.4})$$

The first term in the sum is given by

$$\sum_{p=1}^{\tilde{p}} \frac{2^{-p}}{z + \lambda_\infty - \lambda_p} = c_1 z^{\frac{2-\tau}{\tau-1}} + \mathcal{O}(z), \quad (\text{A.5})$$

while the second one reads

$$\sum_{p=\tilde{p}+1}^{\infty} \frac{2^{-p}}{z + \lambda_\infty - \lambda_p} \leq c_2 z^{\frac{2-\tau}{\tau-1}}. \quad (\text{A.6})$$

Thus, for $\tau > 2$ and $z \rightarrow 0$, we get

$$g'(z) \sim A z^{\frac{2-\tau}{\tau-1}}, \quad (\text{A.7})$$

with c_1 , c_2 and A representing positive constants.

Let us now consider $g''(z)$

$$g''(z) = -\frac{1}{2} \sum_{p=1}^{\infty} \frac{2^{-p}}{(z + \lambda_\infty - \lambda_p)^2}. \quad (\text{A.8})$$

For $\tau < 3/2$ the series is convergent. In particular, we have

$$g''(0) = -\frac{1}{2^{2\tau-1}(\lambda_\infty - \lambda_1)^2} \left[\frac{1}{1 - 2^{-(3-2\tau)}} - 1 \right]. \quad (\text{A.9})$$

As before, we want to study the case of $\tau > 3/2$, with z positive and small. The function $g''(z)$ is always convergent for $z > 0$, but we expect a divergence as $z \rightarrow 0$ when $\tau > 3/2$. By repeating the same argument as for $g'(z)$, in this situation we derive the expression

$$g''(z) \sim B z^{\frac{3-2\tau}{\tau-1}}. \quad (\text{A.10})$$

On the same lines we can get the behaviour of $g'''(z)$

$$g'''(z) = \sum_{p=1}^{\infty} \frac{2^{-p}}{(z + \lambda_\infty - \lambda_1)^3}. \quad (\text{A.11})$$

The above series is always convergent. For $\tau < 4/3$, we have a finite value for $g'''(0)$, while we obtain that, for $z \rightarrow 0$ and $\tau > 4/3$, the behaviour of $g'''(z)$ is given by

$$g'''(z) \sim C z^{\frac{4-3\tau}{\tau-1}}. \quad (\text{A.12})$$

To summarise, we have the following asymptotic behaviours

- $\lim_{z \rightarrow 0} g'(z) \sim z^{\frac{2-\tau}{\tau-1}}$ for $\tau > 2$, while it is finite for $\tau < 2$;
- $\lim_{z \rightarrow 0} g''(z) \sim z^{\frac{3-2\tau}{\tau-1}}$ for $\tau > \frac{3}{2}$, while it is finite for $\tau < \frac{3}{2}$;
- $\lim_{z \rightarrow 0} g'''(z) \sim z^{\frac{4-3\tau}{\tau-1}}$ for $\tau > \frac{4}{3}$, while it is finite for $\tau < \frac{4}{3}$.

References

- [1] Y. Imry and S.-K. Ma, “Random-field instability of the ordered state of continuous symmetry,” *Physical Review Letters*, vol. 35, pp. 1399–1401, 1975.
- [2] A. Aharony, Y. Imry, and S.-K. Ma, “Lowering of dimensionality in phase transitions with random fields,” *Physical Review Letters*, vol. 37, pp. 1364–1367, 1976.
- [3] A. Young, “On the lowering of dimensionality in phase transitions with random fields,” *Journal of Physics C: Solid State Physics*, vol. 10, no. 9, p. L257, 1977.
- [4] G. Parisi and N. Sourlas, “Random magnetic fields, supersymmetry, and negative dimensions,” *Phys. Rev. Lett.*, vol. 43, pp. 744–745, 1979.
- [5] J. Z. Imbrie, “Rounding of first-order phase transitions in systems with quenched disorder,” *Phys. Rev. Lett.*, vol. 53, p. 1747, 1984.
- [6] G. Tarjus, I. Balog, and M. Tissier, “Critical scaling in random-field systems: 2 or 3 independent exponents?,” *EPL (Europhysics Letters)*, vol. 103, no. 6, p. 61001, 2013.
- [7] H. De Dominicis, C. Orland and T. Temesvari, “Random field ising model: dimensional reduction or spin-glass phase?,” *Journal de Physique I*, vol. 5, no. 8, pp. 987–1001, 1995.
- [8] F. Krzakala, F. Ricci-Tersenghi and L. Zdeborová, “Elusive spin-glass phase in the random field ising model,” *Phys. Rev. Lett.*, vol. 104, no. 20, p. 207208, 2010.
- [9] F. Krzakala, F. Ricci-Tersenghi, D. Sherrington, and L. Zdeborová, “No spin glass phase in the ferromagnetic random-field random-temperature scalar ginzburg-landau model,” *J. Phys. A*, vol. 44, no. 4, p. 042003, 2011.
- [10] F. J. Dyson, “Existence of a phase-transition in a one-dimensional ising ferromagnet,” *Comm. Mat. Phys.*, vol. 12, no. 2, p. 91, 1969.
- [11] G. A. Baker Jr, “Ising model with a scaling interaction,” *Phys. Rev. B*, vol. 5, no. 7, p. 2622, 1972.
- [12] G. A. Baker Jr and G. R. Golner, “Spin-spin correlations in an ising model for which scaling is exact,” *Phys. Rev. Lett.*, vol. 31, no. 1, p. 22, 1973.
- [13] P. Collet and J. P. Eckmann, *The RG-transformation for the hierarchical model*. Springer, 1978.
- [14] K. G. Wilson and J. Kogut, “The renormalization group and the ϵ expansion,” *Phys. Rep.*, vol. 12, no. 2, p. 75, 1974.
- [15] G. Felder, “Renormalization group in the local potential approximation,” *Communications in Mathematical Physics*, vol. 111, no. 1, pp. 101–121, 1987.
- [16] D. Kim and C. J. Thompson, “Critical properties of dyson’s hierarchical model,” *J. Phys. A*, vol. 10, no. 9, p. 1579, 1977.
- [17] T. Franz, S. Jörg and G. Parisi, “Overlap interfaces in hierarchical spin-glass models,” *J. Stat. Mec.*, vol. 2009, no. 02, p. P02002, 2009.
- [18] L. Leuzzi, G. Parisi, F. Ricci-Tersenghi, and J. Ruiz-Lorenzo, “Dilute one-dimensional spin glasses with power law decaying interactions,” *Physical review letters*, vol. 101, no. 10, p. 107203, 2008.
- [19] H. G. Katzgraber, D. Larson, and A. Young, “Study of the de almeida-thouless line using power-law diluted one-dimensional ising spin glasses,” *Physical review letters*, vol. 102, no. 17, p. 177205, 2009.
- [20] M. Castellana, A. Decelle, S. Franz, M. Mézard, and G. Parisi, “Hierarchical random energy model of a spin glass,” *Phys. Rev. Lett.*, vol. 104, no. 12, p. 127206, 2010.
- [21] M. Castellana and G. Parisi, “Renormalization group computation of the critical exponents of hierarchical spin glasses,” *Phys. Rev. E*, vol. 82, no. 4, p. 040105, 2010.
- [22] M. Castellana, “Real-space renormalization group analysis of a non-mean-field spin-glass,” *EPL*, vol. 95, no. 4, p. 47014, 2011.
- [23] M. C. Angelini, G. Parisi, and F. Ricci-Tersenghi, “Ensemble renormalization group for disordered systems,” *Phys. Rev. B*, vol. 87, no. 13, p. 134201, 2013.
- [24] C. Monthus and T. Garel, “Random field ising model: statistical properties of low-energy excitations and equilibrium avalanches,” *J. Stat. Mec.*, vol. 2011, no. 07, p. P07010, 2011.
- [25] L. A. Baños, R. A. Fernandez, V. Martín-Mayor, and A. P. Young, “Correspondence between

- long-range and short-range spin glasses,” *Phys. Rev. B*, vol. 86, no. 13, p. 134416, 2012.
- [26] L. Leuzzi and G. Parisi, “Imry-ma criterion for long-range random field ising model: short-/long-range equivalence in a field,” *arXiv preprint arXiv:1303.6333*, 2013.
 - [27] G. Parisi and J. Rocchi, “Critical exponents of the random field hierarchical model,” *arXiv preprint arXiv:1309.7470*, 2013.
 - [28] A. Decelle, G. Parisi, and J. Rocchi, “Ensemble renormalization group for the random-field hierarchical model,” *Phys. Rev. E*, vol. 89, p. 032132, Mar 2014.
 - [29] M. Angelini, G. Parisi, and F. Ricci-Tersenghi, “Relations between short range and long range ising models,” *Preprint at arXiv:1401.6805*, 2014.
 - [30] T. Berlin and M. Kac, “The spherical model of a ferromagnet,” *Physical Review*, vol. 86, no. 6, p. 821, 1952.
 - [31] J. McGuire, “The spherical hierarchical model,” *Communications in Mathematical Physics*, vol. 32, no. 3, pp. 215–230, 1973.
 - [32] G. Ben Arous, O. Hryniv, and S. Molchanov, “Phase transition for the spherical hierarchical model,” *Markov processes and related fields.*, vol. 8, no. 4, pp. 565–594, 2002.
 - [33] R. Baxter, *Exactly solved models in statistical mechanics*. Academic Press, 1982.
 - [34] R. Hornreich and H. Schuster, “Thermodynamic properties of the random-field spherical model,” *Physical Review B*, vol. 26, no. 7, p. 3929, 1982.
 - [35] C. De Dominicis and I. Giardinà, *Random Fields and Spin Glasses: A Field Theory Approach*. Cambridge: Cambridge Univ. Press, 2006.
 - [36] J. Rocchi, “The renormalization group and the random field model,” *Thesis*, 2014.
 - [37] C. Monthus and T. Garel, “A critical dyson hierarchical model for the anderson localization transition,” *JSTAT*, p. P05005, 2011.
 - [38] F. L. Metz, L. Leuzzi, G. Parisi, and V. Sacksteder, “Transition between localized and extended states in the hierarchical anderson model,” *Phys. Rev. B*, vol. 88, p. 045103, Jul 2013.
 - [39] F. L. Metz, L. Leuzzi, and G. Parisi, “Renormalization flow of the hierarchical anderson model at weak disorder,” *Phys. Rev. B*, vol. 89, p. 064201, Feb 2014.
 - [40] J. R. L. de Almeida and D. J. Thouless, “Stability of the sherrington-kirkpatrick solution of a spin glass model,” *Journal of Physics A: Mathematical and General*, vol. 11, no. 5, p. 983, 1978.
 - [41] A. J. Bray and M. A. Moore, “Scaling theory of the random-field ising model,” *J. Phys. C*, vol. 18, no. 28, p. L927, 1985.
 - [42] G. J. Rodgers and A. J. Bray, “Critical behaviour of dyson’s hierarchical model with a random field,” *J. Phys. A*, vol. 21, no. 9, p. 2177, 1988.
 - [43] H. Stanley, “Spherical model as the limit of infinite spin dimensionality,” *Physical Review*, vol. 176, no. 2, p. 718, 1968.